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Univariate and bivariate Burr x-type distributions

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ABSTRACT

The Burr X distribution has been extensively studied by many researchers. It has many applications in medical, biological, agriculture and other fields. In this paper, a new family of Burr X-type distributions is introduced; the univariate Burr X-type distribution and the bivariate Burr X-type distribution. The bivariate Burr X-type distribution is constructed based on Gaussian copula with univariate Burr X-type distribution as marginals. This type distribution is more flexible and provides easier implementation and extension to bivariate form. A Gibbs sampler procedure is used to obtain Bayesian estimates of the unknown parameters. A simulation study is carried out to illustrate the efficiency of the proposed bivariate Burr X-type distribution. Finally, the proposed bivariate distribution is applied on real data to demonstrate its usefulness for real life applications.

1. Introduction

Burr type *X* distribution is a member of the family of Burr distributions which was appeared since 1942 (Burr, 1942). It is known also as generalized Rayleigh distribution. This distribution has increasing importance in several areas of applications such as lifetime tests, health, agriculture, biology, and other sciences.

In recent years, Burr X distribution has been extensively used in medical, biological, agriculture, lifetime tests, and other sciences applications. The Burr X distribution was first introduced by Burr (1942) and later a generalized form of this distribution is introduced by Mudholkar and Srivastava (1993). Several characteristics and inferences of this distribution were studied by many researchers, see for example Abd et al. (2015), Ali Mousa (2001), Aludaat et al. (2008), Jaheen and Al-Matrafi (2002), Kjelsberg (1962), Kundu and Raqab (2005) and Raqab (1998), among others. The probability density function (Pdf) of the Burr type X distribution with shape parameter β and scale parameter α is given by

$$f(t) = \frac{2\beta t}{\alpha^2} \exp\left(-\left(\frac{t}{\alpha}\right)^2\right)\left[1-\exp\left(-\left(\frac{t}{\alpha}\right)^2\right)\right]^{\beta-1}, \quad (1)$$

where t, α, β >0.

Diverse methods have been studied in previous research for establishing new multivariate or bivariate distributions. Among these, the study of Al-Hussaini and Ateya (2005), Johnson et al. (2002), Marshall and Olkin (1997) and Walker and Stephens (1999). Recently, the copula method has received special attention for constructing multivariate or bivariate distributions due to its simplicity and useful dependency properties. Some of these studies combined the mixture and copula ideas to establish new family of distributions which concluded that the resulting multivariate or bivariate distribution is easy to analyze and has a full dependence structures. These studies include Adham and Walker (2001), AL Dayian et al. (2008), Abd Elaal et al. (2016), and Adham et al. (2009).

According to Adham and Walker (2001) and Walker and Stephens (1999), the mixture representation for a Pdf of a random variable Ton [0,∞) can be written in the following form

$$f(t) = \int_\Omega f(t|u)\, f(u)\, du, \quad \text{for all } u \in \Omega, \quad (2)$$

where u is a non-negtive latent variable that follows a gamma distribution with shape parameter 2 and scale parameter 1. Then, the mixture representation for any lifetime distribution can be written as

$$f(t) = \int_\Omega f(t|u)\, f(u)\, du, \quad \text{for all } u \in \Omega, \quad (3)$$

where $h(t)$ and $H(t)$ are the hazard rate function and the cumulative hazard rate function of T, respectively.

The studies of Walker and Stephens (1999), Agarwal and Al-Saleh (2001), and Arslan (2005)

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have suggested type distributions by replacing the gamma mixing distribution with lognormal distribution which extend more easily to multivariate form. In this paper, we will apply the mixture and copula on Burr X-type distribution to introduce the bivariate Burr X-type distribution.

Therefore, we replace the gamma mixing distribution in (3) by lognormal distribution to obtain the Burr X-type distribution. However, the normal distribution with mean $=\mu$ and variance $=\sigma^2$ denoted by $N(\mu,\sigma^2)$, will be used for simplicity after considering the appropriate transformation. Thus, equation (3) can be rewritten as

$$f(t|u)=h(t)\exp(-u),u>\ln(H(t)),u\sim N(\mu,\sigma^2). \quad (4)$$

The paper is outlined as follows: Section 2 deals with the construction and estimation for the unknown parameters of the univariate Burr X-type distribution. Section 3 presents the construction of the new bivariate Burr X-type distribution. Bayesian estimation of the parameters of the bivariate Burr X-type is discussed in Section 4. Simulation study is carried out in Section 5 to illustrate the performance of the proposed bivariate Burr X-type distribution. Finally, real data application is analyzed in Section 6 to show the flexibility of the bivariate Burr X-type distribution.

2. The univariate Burr X-type distribution

The hazard rate function (HRF) and the cumulative hazard rate function (CHRF) of a continues random variable T that follows a Burr X distribution are given, respectively, by

$$h(t)=\frac{\frac{2t\beta}{\alpha^2}(1-\omega)\omega^{\beta-1}}{1-\omega^\beta} \quad (5)$$

and

$$H(t)=-\ln(1-\omega^\beta), \quad (6)$$

where $\omega=1-\exp\left(-\left(\frac{t}{\alpha}\right)^2\right)$, $t,\alpha,\beta>0$.

The mixture representation for the Burr X-type is obtained by substituting (5) and (6) in (4) as follows

$$f(t|u)=\frac{\frac{2t\beta}{\alpha^2}(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}exp(-u),u>ln(H(t)) \quad (7)$$

Therefore, the Pdf of Burr X-type can be written as

$$f(t)=\frac{\frac{2t\beta}{\alpha^2}(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}\exp\left(\frac{\sigma^2}{2}-\mu\right)\left[1-\Phi\left(\frac{G-(\mu-\sigma^2)}{\sigma}\right)\right] \quad (8)$$

where $t,\sigma^2>0,-\infty<\mu<\infty,\Phi$, is the distribution function of the standard normal distribution, and $G=ln(H(t))$.

2.1. Parameters estimation of the univariate Burr X-type distribution

Parameter estimates of the Burr X-type are obtained using Bayesian estimation. The Gibbs sampler will be used to obtain random variables from posterior distributions of the Burr X-type distribution, see Adham and Walker (2001), Gilks and Wild (1992), and Gilks et al. (1995).

Let $T=t_1,\dots,t_n$ be a random sample from Burr X distribution, and $U=u_1,\dots,u_n$ is a random sample from $N(\mu,\sigma^2)$ distribution. Then, the likelihood function can be written as

$$L(\alpha,\beta|T,U)=\left(\frac{2\beta}{\alpha^2}\right)^n exp\left[\sum_{i=1}^n ln\left(\frac{t_i(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}\right)\right]-\left[\frac{1}{2\sigma^2}\sum_{i=1}^n\left(u_i-(\mu-\sigma^2)\right)^2\right] \quad (9)$$

Non-informative priors for the parameter α and β are given by

$$\pi(\alpha)=\frac{1}{\alpha},\qquad \pi(\beta)=\frac{1}{\beta}. \quad (10)$$

Then, the posterior distribution is

$$f(\alpha,\beta|T,U)=\frac{\beta^{n-1}}{\alpha^{2n+1}}exp\left[\sum_{i=1}^n ln\left(\frac{t_i(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}\right)\right]-\left[\frac{1}{2\sigma^2}\sum_{i=1}^n\left(u_i-(\mu-\sigma^2)\right)^2\right]. \quad (11)$$

It can be seen that the above conditional distribution of the parameters is not in closed form and we need to apply Monte Carlo integration by sampling the full conditional distributions of the parameters as follows:

First: The marginal posterior of u_i, i=1,…,n

$$f(u_i|\alpha,\beta,t)=\frac{1}{2\sigma^2}\sum_{i=1}^n\left(u_i-(\mu-\sigma^2)\right)^2, \quad (12)$$

where $u>ln\left(-ln\left(1-\omega^\beta\right)\right)$, which is a left truncated normal distribution, see Robert (1995).

Second: The marginal posterior of α

$$f(\alpha|\beta,T,U)\propto\frac{1}{\alpha^{2n+1}}exp\left[\sum_{i=1}^n ln\left(\frac{t_i(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}\right)\right], \quad (13)$$

where $\alpha>\frac{t}{\sqrt{ln((1-exp(-u))^{1/\beta}-1)}}$.

Third: The marginal posterior of the parameter β

$$f(\beta|\alpha,T,U)\propto\beta^{n-1}exp\left[\sum_{i=1}^n ln\left(\frac{t_i(1-\omega)\omega^{\beta-1}}{1-\omega^\beta}\right)\right], \quad (14)$$

where $\beta>\frac{ln(1-exp(-u))}{ln(\omega)}$.

These above marginal posteriors of α and β are not in closed form. Therefore, Algorithms 1 and 2 in Appendices A and B are applied to sample the posterior distribution of the parameters α and β, respectively.

3. Bivariate Burr X-type distribution

The construction of bivariate Burr X-type distribution is based on copula and M mixture representation, where M denotes the set of densities for a random variable T on [0, ∞).

Suppose we have two-dimensional random vectors $\underline{T} = (T_1, T_2)$ and $\underline{U} = (U_1, U_2)$, where $\underline{U} \sim BVN\left(\underline{\mu}, \Sigma\right)$, $\underline{\mu} = (\mu_1, \mu_2)$, $\Sigma = \lfloor \sigma_{Lj} \rfloor_{2x2}$ for j=1, 2, and Σ is the variance covariance matrix.

Assuming that T_1, T_2 are conditionally independent given U_1, U_2, the joint conditional pdf of $\underline{T}$ on $\underline{U}$ is given by

$$f(\underline{T}|\underline{U}) = \prod_{j=1}^{2} f(t_j|u_j) = \prod_{j=1}^{2} h(t_j)\exp(-u_j). \qquad (15)$$

Then, the joint pdf of the bivariate Burr X-type based on M mixture representation is

$$f(\underline{T}) = \int_{G_1}^{\infty}\int_{G_2}^{\infty} f(t_j|u_j) f(u_j)\, du_1\, du_2 \;, \qquad (16)$$

where $G_{j=}\ln(H(t_j))$, $f(t_j|u_j)$ is given by (7) and $f(u_j), j = 1,2$ is the bivariate normal density function, denoted by BVN $\left(\underline{\mu}, \Sigma\right)$. The joint pdf of $\underline{T}$ cannot be written in closed form except in terms of the two-dimensional standard bivariate normal distribution function. However, this does not cause any problem in examining the distribution.

4. Parameters estimation of the bivariate Burr X-type distribution

The vector of parameters of the bivariate Burr X-type, $(\alpha_1, \alpha_2, \beta_1, \beta_2, \rho)$, where ρ is the correlation parameter of the bivariate normal distribution is estimated using Bayesian method. Considering that the prior distribution of the parameters $\alpha_j$andγ_j are

$$\pi(\alpha_j) \propto \frac{1}{\alpha_j}, \quad \pi(\gamma_j) \propto \frac{1}{\gamma_j}, \quad j = 1,2. \qquad (17)$$

And the correlation parameter ρ has a uniform prior distribution defined on the interval (-1, 1). Therefore, the likelihood function can be rewritten as

$$L\left(\underline{\alpha}, \underline{\beta}\middle|\underline{T}, \underline{U}\right) = \prod_{j=1}^{2}\left(\frac{2\beta_j}{\alpha_j^2}\right)^n exp\left[\sum_{i=1}^{n} ln\left(\frac{t_{ji}(1-\omega_j)\omega_j^{\beta_j-1}}{1-\omega_j^{\beta_j}}\right)(1-\rho^2)^{-\frac{n}{2}} exp\left(\frac{-1}{\sigma^2(1-\rho^2)}\sum_{i=1}^{n}\xi_i\right)\right], \qquad (18)$$

where

$$\xi_i = (u_{1i} - \mu_1)^2 - 2\rho(u_{1i} - \mu_1)(u_{2i} - \mu_2) + (u_{2i} - \mu_2)^2, \underline{\alpha} = (\alpha_1, \alpha_2), \underline{\beta} = (\beta_1, \beta_2) \qquad (19)$$

The joint posterior distribution of the parameters given a random sample of size n from the bivariate Burr X-type is given by

$$f\left(\underline{\alpha}, \underline{\beta}, \rho\middle|\underline{T}, \underline{U}\right) = \prod_{j=1}^{2}\left(\frac{\beta_j}{\alpha_j^2}\right)^n exp\left[\sum_{i=1}^{n} ln\left(\frac{t_{ji}(1-\omega_j)\omega_j^{\beta_j-1}}{1-\omega_j^{\beta_j}}\right)(1-\rho^2)^{-\frac{n}{2}} exp\left(\frac{-1}{\sigma^2(1-\rho^2)}\sum_{i=1}^{n}\xi_i\right)\right]. \qquad (20)$$

Then, we sample the following conditional distributions:

First: Sample the marginal posterior of u_{ji} from

$$f(u_{ji}|u_{\varepsilon i}, t_i, \alpha, \beta, \rho) \sim Normal\left(\mu_j + \rho(u - \mu_\varepsilon) - \sigma^2(1-\rho^2), \sigma^2(1-\rho^2)\right).$$

This is restricted to the interval

$\left(\ln\left(-\ln\left[1 - \omega_j^{\beta_j}\right]\right), \infty\right)$, for ε =1,2, ε ≠ j.

Second: Sample the marginal posterior of α_j from

$$f(\alpha_j|\alpha_\varepsilon, \beta_j, \rho, \underline{T}, \underline{U}) \propto \frac{1}{\alpha_j^{2n+1}} \exp\left[\sum_{i=1}^{n} \ln\left(\frac{t_{ji}(1-\omega_j)\omega_j^{\beta_j-1}}{1-\omega_j^{\beta_j}}\right)\right], \qquad (21)$$

where $\alpha_j > \frac{t_{ji}}{\sqrt{ln\left((1-\exp(-u_{ji}))^{1/\beta_j}-1\right)}}$. Algorithm 1 is used to sample this conditional distribution.

Third: Sample the marginal posterior of β_j from

$$f(\beta_j|\beta_\varepsilon, \alpha_j, \rho, \underline{T}, \underline{U}) \propto \beta_j^{n-1} exp\left[\sum_{i=1}^{n} ln\left(\frac{t_{ji}(1-\omega_j)\omega_j^{\beta_j-1}}{1-\omega_j^{\beta_j}}\right)\right], \qquad (22)$$

where $\beta_j > \frac{\ln(1-\exp(-u_{ji}))}{\ln(\omega_j)}$. We apply Algorithm 2 to sample this full conditional distribution.

Finally, sample ρ from its posterior distribution

$$f\left(\rho\middle|\underline{\alpha}, \underline{\beta}, T, \underline{U}\right) = (1-\rho^2)^{-\frac{n}{2}} \exp\left(\frac{-1}{\sigma^2(1-\rho^2)}\sum_{i=1}^{n}\xi_i\right) \qquad (23)$$

This full conditional distribution can be sampled using metropolis Hasting Algorithm.

5. Simulation study

In this Section, simulation study is carried out to examine the performance of the Bayesian estimation for different sample sizes and parameter values for the constructed bivariate Burr X-type distribution. The performances of the Bayesian estimates are studied mainly with respect to the mean squared error (MSE) over 1000 iterations. Random samples of sizes (n=15, 30, 50) observations are generated from the bivariate Burr X-type distribution with marginal distributions BurrX-type $(\alpha_1 = 0.3,\ \beta_1 = 0.7)$ and BurrX-type $(\alpha_2 = 0.5,\ \beta_2 = 1.1)$ with ρ=0.66. Moreover, another random samples of sizes (n=15, 30, 50) observations are generated from the bivariate Burr X-type distribution with marginal distributions BurrX-type$(\alpha_1 = 2,\ \beta_1 = 2.5)$ and BurrX-type$(\alpha_2 = 0.5,\ \beta_2 = 1.8)$ with ρ=0.7 and results of the simulation are reported in Tables 1 and 2.

The predicted observations can be obtained by applying the inverse of the conditional distribution function of the BurrX -type distribution as

$$t_{jk} = \alpha_{jk}\sqrt{ln\left(\left(1-\left(1-exp\left(-\frac{v_{jk}}{\exp(-u_{jk})}\right)\right)^{\frac{1}{\beta_{jk}}}\right)^{-1}\right)}, \qquad (24)$$

where for j=1,2, v~uniform (0, 1), α_{jk}, β_{jk} are the sampled values of α_j and β_j, respectively, at iteration k of the MCMC run and u_{jk} is the jth element of a new bivariate vector of the observations generated from the bivariate normal mixing distribution with correlation parameter which is sampled at the kth iteration of the MCMC.

Table 1: Bayesian estimate, Bias, variance, and MSE of the bivariate Burr X-type distribution parameters for ρ=0.66

$\alpha_1 = 0.3, \alpha_2 = 0.5, \beta_1 = 0.7, \beta_2 = 1.1, \rho = 0.66$						
n		$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
15	Estimate	0.3099	0.4629	0.7979	1.1685	0.7710
	Biase	0.0099	0.0369	0.0979	0.0685	0.1110
	Variance	6.9e-6	0.0004	0.0172	0.0638	0.0008
	MSE	0.0001	0.0017	0.0269	0.0838	0.0132
30	Estimate	0.3049	0.4582	0.7787	1.0770	0.7715
	Biase	0.0049	0.0417	0.0787	0.0226	0.1115
	Variance	7.8e-07	0.0002	0.0069	0.01941	0.0010
	MSE	2.5e-05	0.0019	0.0131	0.01992	0.0134
50	Estimate	0.3029	0.45667	0.7727	1.0646	0.7652
	Biase	0.0029	0.04332	0.0727	0.0354	0.1052
	Variance	2.0e-07	0.0001	0.0037	0.0092	0.0006
	MSE	9.2e-06	0.001978	0.0090	0.0104	0.0117

Table 2: Bayesian estimate, Bias, variance, and MSE of the bivariate Burr X-type distribution parameters for ρ=0.7

$\alpha_1 = 2, \alpha_2 = 0.5, \beta_1 = 2.5, \beta_2 = 1.8, \rho = 0.7$						
n		$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}$
15	Estimate	2.0651	0.5181	2.1584	1.6068	0.7429
	Biase	0.0651	0.0181	0.3415	0.1931	0.0429
	Variance	0.0003	2.6e-05	0.1326	0.0965	3.9e-07
	MSE	0.0046	0.0003	0.1326	0.1338	0.0018
	Estimate	2.0325	0.5088	2.0700	1.5718	0.7719
	Biase	0.0325	0.0088	0.4299	0.2282	0.0719
	Variance	3.8e-05	2.7e-06	0.0449	0.0290	1.7e-07
	MSE	0.0011	7.9e-05	0.2298	0.0811	0.0052
50	Estimate	2.0199	0.50513	2.0547	1.5662	0.7768
	Biase	0.0199	0.0051	0.4453	0.2338	0.0769
	Variance	7.9e-06	5.3e-07	0.0250	0.0157	9.5e-08
	MSE	0.0004	2.7e-05	0.2233	0.0704	0.0059

As expected, we observe from the results in Table 1 and Table 2 that for all selected values of α_1, α_2 β_1, β_2 and ρ, the MSE of the estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\rho}$ become smaller as the sample size increases. That is, the results improve with increases in sample size. The bias, variance and MSE of the $\hat{\alpha}_1$, $\hat{\alpha}_2$ are smaller compared to $\hat{\beta}_1$, $\hat{\beta}_2$ for all sample sizes. It is observed that the value of p have minimal effects on the estimate of the other parameters.

6. Data analysis

The real dataset is taken from the American football league and provides two game times of the matches played in 1986; the game time to the first points scored by kicking the ball between goalposts denoted by $T_1$and the game time to the first points scored by moving the ball into the end zone denoted by T_2, for details see Csörgő and Welsh (1989). Since these times are positively correlated and their plots are skewed, bivariate Burr X-type distribution can be used to model the data. Kolmogorov-Smirnov (KS) is conducted on the marginal to examine whether the bivariate Burr X-type distribution fits the data (Kundu and Gupta, 2011). The KS test statistics and p-value for $\mathbf{T_1}$ are 0.1419 (0.3666) and for $\mathbf{T_2}$ are 0.1525 (0.2557). Therefore, the bivariate Burr X-type distribution provides suitable fit for the bivariate data. In addition, Al-Urwi and Baharith (2017) showed that the Gaussian copula is appropriate for this data and the inversion of Kendall's tau estimated the copula parameter p to be 0.88 which can be set as initial value when fitting bivariate Burr X-type distribution (Genest et al., 2009). Bayesian estimates, standard error (SE), and credible intervals of the bivariate Burr X-type parameters are reported in Table 3.

Table 3: Bayesian estimates, SE, and the corresponding 95% credible interval of the bivariate Burr X-type distribution parameters

Parameter	estimate	SE	95% Credible Interval	
			2.5%	97.5%
$\hat{\alpha}_1$	0.0170	0.0065	0.0063	0.0313
$\hat{\alpha}_2$	0.0174	0.0052	0.0095	0.0174
$\hat{\beta}_1$	0.2939	0.0255	0.2435	0.3391
$\hat{\beta}_2$	0.2137	0.0187	0.1787	0.2536
$\hat{\rho}$	0.7127	0.0160	0.6822	0.7413

7. Conclusion

In this paper, we propose a family of Burr X-type distributions as a flexible bivariate lifetime distributions which include the univariate Burr X-type distribution and the bivariate Burr X-type distribution. The use of normally distributed latent variables has allowed positive and negative association. Markov chain Monte Carlo simulation is performed to estimate the parameters of the proposed univariate and bivariate distributions. One real lifetime data is analyzed and the results showed the flexibility of the bivariate Burr X-type distribution.

Appendix A. Algorithm 1

1. Given a random sample of $T_{ji} = (T_{1i}, T_{2i})$, $i = 1,\dots,n$ from BurrX distribution, and a random sample of $U_{ji} = (U_{1i}, U_{2i})$, $i = 1,\dots,n$ from normal distribution.
2. Introduce a non-negative latent variable ν_j. The joint probability density function of of v_j and α_j is given by

$$f(\alpha_j, \nu_j) \propto \frac{1}{\alpha_j^{2n}}, \alpha_j > A_j, \nu_j < \frac{d_j}{\alpha_j}, \quad (25)$$

where

$$A_j = \frac{t_{ji}}{\sqrt{\ln\left(\left(1-\exp(-u_{ji})\right)^{1/\gamma_j}-1\right)}} \quad (26)$$

$$d_j = exp\left[\sum_{i=1}^{n} ln\left(\frac{\left(t_{ji}(1-\omega_{ij})\omega_{ij}^{(\gamma_j-1)}\right)}{1-\omega_{ij}^{\gamma_j}}\right)\right] \quad (27)$$

$$\omega_{ij} = 1-\exp\left(-\left(\frac{t_{ij}}{\alpha_j}\right)^2\right) \quad (28)$$

3. Given a value of the parameter α_j, v is sampled from the uniform density on $\left(0, \frac{d_j}{\alpha_j}\right)$.
4. Finally, use the distribution function inverse method to sample α_j

$$f(\alpha_j, \nu) \propto \frac{1}{\alpha_j^{2n}}, A_j < \alpha_j < B_j, B_j < \frac{d_j}{v_{ij}}, \quad (29)$$

$$\alpha_j = \left[A_j^{2n+1} + \delta\left(B_j^{2n+1} - A_j^{2n+1}\right)\right]^{\frac{1}{2n+1}}, \quad (30)$$

where δ is sampled from Uniform$(0,1)$.

Appendix B. Algorithm 2

1. Given a random sample of $T_{ji} = (T_{1i}, T_{2i})$, $i = 1,\dots,n$ from BurrX distribution, and a random sample of $U_{ji} = (U_{1i}, U_{2i})$, $i = 1,\dots,n$ from normal distribution.
2. Introduce a non-negative latent variable ν_j. The joint probability density function of ν_j and γ_j is given by

$$f(\gamma_j, \nu_j) \propto \gamma^{n-2}, \gamma_j < A_j, \nu_j < \gamma_j d_j, \quad (31)$$

where

$$A_j = \frac{\ln(1-\exp(-u)}{ln\left(1-exp\left(-(t_{ji}/\alpha_j)^2\right)\right)}, \quad (32)$$

$$d_j = exp\left[\sum_{i=1}^{n} ln\left(\frac{\left(t_{ji}(1-\omega_{ij})\omega_{ij}^{(\gamma_j-1)}\right)}{1-\omega_{ij}^{\gamma_j}}\right)\right], \quad (33)$$

$$\omega_{ij} = 1-\exp\left(-\left(\frac{t_{ij}}{\alpha_j}\right)^2\right) \quad (34)$$

3. Given a value of the parameter γ_j, ν_j is sampled from the Uniform$(0, \gamma_j d_j)$.

$$f(\gamma_j, \nu_j) \propto \gamma^{n-2}, \quad \mathbf{B}_j < \gamma_j < A_j, \; B_j = \frac{v_j}{d_j}, \quad (35)$$

Then

$$\gamma_j = \left[B_j^{n-1} + \delta\left(A_j^{n-1} - B_j^{n-1}\right)\right]^{\frac{1}{n-1}}, \quad (36)$$

where δ is sampled from Uniform (0, 1).

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